

On the number of solutions of a quadratic equation in a normed space

Victor Alexandrov

Abstract

We study an equation $Qu = g$, where Q is a continuous quadratic operator acting from one normed space to another normed space. Obviously, if u is a solution of such equation then $-u$ is also a solution. We find conditions implying that there are no other solutions and apply them to the study of the Dirichlet boundary value problem for the partial differential equation $u\Delta u = g$. *Mathematics Subject Classification (2010)*: 47H30; 39B52; 35Q60.

Key words: quadratic operator, quadratic functional equation, vector space

§ 1. Continuous quadratic operators and continuous symmetric bilinear operators

Let L and L' be normed spaces over the field of real numbers \mathbb{R} .

Definition 1: A continuous mapping $Q : L \rightarrow L'$ is called a *continuous quadratic operator* if

$$Q(u + v) + Q(u - v) = 2Q(u) + 2Q(v) \quad \text{for all } u, v \in L \quad (1)$$

and

$$Q(ku) = k^2Q(u) \quad \text{for all } u \in L \text{ and all } k \in \mathbb{R}. \quad (2)$$

Definition 2: Given $B : L \times L \rightarrow L'$, a continuous symmetric bilinear mapping, the mapping $Q : L \rightarrow L'$, defined by the formula

$$Q(u) = B(u, u), \quad (3)$$

is called a *continuous quadratic operator*.

Theorem 1: *Definitions 1 and 2 are equivalent.*

Proof: Let Q be as in Definition 1. By definition, put

$$B(u, v) = \frac{1}{4}(Q(u + v) - Q(u - v)). \quad (4)$$

Obviously, B is continuous, Q is generated by B according to the formula (3), and B is symmetric. Less trivial is that B is bilinear. In order to prove the latter, we define mappings $F : L \times L \times L \rightarrow L'$ and $f : \mathbb{R} \times L \times L \rightarrow L'$ by the formulas

$$F(u, v, w) = 4(B(u + v, w) - B(u, w) - B(v, w))$$

and

$$f(k) = B(ku, v) - kB(u, v),$$

where B is defined by (4). Our aim is to prove that

$$F(u, v, w) = 0 \quad \text{for all } u, v, w \in L \quad (5)$$

and

$$f(k, u, v) = 0 \quad \text{for all } u, v \in L \text{ and all } k \in \mathbb{R}. \quad (6)$$

As soon as formulas (5) and (6) will be proved, we can conclude that B is bilinear and, thus, Definition 2 follows from Definition 1.

In order to prove (5), we use (4) and rewrite F in the form

$$F(u, v, w) = Q(u + v + w) - Q(u + v - w) - Q(u + w) + Q(u - w) - Q(v + w) + Q(v - w). \quad (7)$$

It follows from (1) that

$$Q(u + v + w) = 2Q(u + w) + 2Q(v) - Q(u - v + w) \quad (8)$$

and

$$Q(u + v - w) = 2Q(u - w) + 2Q(v) - Q(u - v - w). \quad (9)$$

Substituting (8) and (9) to (7) yields

$$F(u, v, w) = Q(u - v - w) - Q(u - v + w) + Q(u + w) - Q(u - w) + Q(v - w) - Q(v + w). \quad (10)$$

Adding (10) to (7) and taking into account (2), we get $2F(u, v, w) = 0$. Thus (5) is proved.

In order to prove (6), we use (4) and rewrite f in the form

$$f(k, u, v) = \frac{1}{4}(Q(ku + v) - Q(ku - v)) - \frac{1}{4}(Q(u + v) - Q(u - v)). \quad (11)$$

From (11), we find $f(0, u, v) = 0$ and, using (2),

$$f(-1, u, v) = 0. \quad (12)$$

Given any integer $n \neq 0$, we use (5) and (12) to obtain

$$\begin{aligned} f(n, u, v) &= B(nu, v) - nB(u, v) = B(\operatorname{sgn} n(u + u + \cdots + u), v) - nB(u, v) \\ &= \operatorname{sgn} n(B(u, v) + B(u, v) + \cdots + B(u, v)) - nB(u, v) = nB(u, v) - nB(u, v) = 0. \end{aligned}$$

Here $\operatorname{sgn} n$ denotes the sign of n , i.e., $\operatorname{sgn} n = +1$ for $n > 0$, $\operatorname{sgn} n = 0$ for $n = 0$, and $\operatorname{sgn} n = -1$ for $n < 0$.

If k is a rational number, $k = m/n$, then

$$\begin{aligned} f(k, u, v) &= B\left(\frac{m}{n}u, v\right) - \frac{m}{n}B(u, v) = mB\left(\frac{1}{n}u, v\right) - \frac{m}{n}B(u, v) \\ &= \frac{m}{n}nB\left(\frac{1}{n}u, v\right) - \frac{m}{n}B(u, v) = \frac{m}{n}B(u, v) - \frac{m}{n}B(u, v) = 0. \end{aligned}$$

Thus, $f(k, u, v) = 0$ for all rational k and all $u, v \in L$. Since f is continuous, $f(k, u, v) = 0$ for all $k \in \mathbb{R}$ and all $u, v \in L$. This completes the proof of (6). Hence, if Q satisfies Definition 2, it satisfies Definition 1 also.

At last, observe that straightforward computations show that, if Q satisfies Definition 1, it satisfies Definition 2 also. Theorem 1 is proved.

Theorem 1 shows that there is a one-to-one correspondence between continuous quadratic operators $Q : L \rightarrow L'$ and continuous symmetric bilinear operators $B : L \times L \rightarrow L'$. By B_Q we denote the unique continuous symmetric bilinear operator that corresponds to a continuous quadratic operator Q . Thus, $Q(u) = B_Q(u, u)$ for all $u \in L$.

Quadratic operators were studied by many authors from different points of view. For more details the reader is referred, for example, to [1] and references given therein. We are going to study a quadratic equation $Qu = g$, where $Q : L \rightarrow L'$ is a continuous quadratic operator acting from a normed space L to another normed space L' . It follows from (2) that if u is a solution of a quadratic equation then $-u$ is also a solution. We are interested in the following question: given a continuous quadratic operator Q , is it true that, for all g , the equation $Qu = g$ has no more than two solutions? The answer is given by the following

Theorem 2: *Let L and L' be normed spaces over the field \mathbb{R} of real numbers, $Q : L \rightarrow L'$ be a continuous quadratic operator. Then the following conditions are equivalent:*

(i) *for all $g \in L'$, the equation $Qu = g$ has no more than two solutions, i.e., $Q(u) = Q(v)$ implies $v = \pm u$;*

(ii) *B_Q is nondegenerate, i.e., for every $u \in L$, $u \neq 0$, the equality $B_Q(u, v) = 0$ implies $v = 0$.*

Proof: Suppose that condition (ii) is not satisfied. This means that, in L , there exist $u \neq 0$ and $v \neq 0$ such that $B(u, v) = 0$. Then, according to (4), $Q(u + v) - Q(u - v) = 4B_Q(u, v) = 0$. On the other hand, $u + v \neq u - v$ as well as $u + v \neq -(u - v)$. Hence, condition (i) is not satisfied. In other words, this means that condition (i) implies condition (ii).

Now suppose that condition (i) is not satisfied. This means that there are two vectors $u, v \in L$ such that $Q(u) = Q(v)$ and neither $v = u$ nor $v = -u$. By definition, put $U = \frac{1}{2}(u + v)$ and $V = \frac{1}{2}(u - v)$. Obviously, $U \neq 0$ and $V \neq 0$. Moreover, according to (4), $4B_Q(U, V) = Q(U + V) - Q(U - V) = Q(u) - Q(v) = 0$. Hence, condition (ii) is not satisfied. In other words, condition (ii) implies condition (i). Theorem 2 is proved.

§ 2. On the number of solutions to a boundary value problem for the partial differential equation $u\Delta u = g$

In order to demonstrate the advantages of the replacement of the quadratic equation $Qu = g$ by the linear equation $B_Q(u, v) = 0$ (g and v are prescribed functions) described in the previous section, we consider the following Dirichlet problem:

$$u(x, y) \left(\frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) \right) = g(x, y) \quad \text{for } (x, y) \in D, \quad (13)$$

$$u(x, y) = 0 \quad \text{for } (x, y) \in \partial D, \quad (14)$$

where

$$D = \left\{ (x, y) \in \mathbb{R}^2 \mid -\frac{\pi}{2} < x < \frac{\pi}{2}, -\frac{\pi}{2} < y < \frac{\pi}{2} \right\}$$

is a square and ∂D is its boundary.

Obviously, if u is a solution to (13)–(14) then $-u$ is also a solution. We are interested if, for every f , the problem (13)–(14) has no more than two solutions. Equivalently, our problem can be reformulated as follows: is it true that if u and v are two solutions to (13)–(14) then $v = \pm u$?

In this section we prove that the answer is negative.

The nonlinear PDE operator in the left-hand side of (13) is a simplification of the operator in the equation $u\Delta u + C_1|\nabla u|^2 = C_2$, where $C_1, C_2 \in \mathbb{R}$, for which a strong minimum principle is proved among other results in [2].

Let L be the linear space of functions $u : D \rightarrow \mathbb{R}$, each of which is of the class C^∞ in some (specific for every u) open set containing the closure of D and vanishes on ∂D , endowed with the norm $\|u\|_L = \sup_{(x,y) \in D} |u(x, y)| + \sup_{(x,y) \in D} |\nabla u(x, y)|$, where

$$\nabla u(x, y) = \left(\frac{\partial u}{\partial x}(x, y), \frac{\partial u}{\partial y}(x, y) \right)$$

is the gradient of u . Let L' be the linear space of functions which are continuous in the closure of D and are endowed with the norm $\|u\|_{L'} = \sup_{(x,y) \in D} |u(x,y)|$. At last, let the operator $Q : L \rightarrow L'$ be defined by the formula $Qu = u\Delta u$ or, equivalently,

$$(Qu)(x,y) = u(x,y) \left(\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) \right).$$

Obviously, Q is a continuous quadratic operator from L to L' for which $B_Q(u,v) = u\Delta v + v\Delta u$ and the equation $Qu = g$ coincides with (13).

Theorem 3: *Let D , L , Q , and B_Q be as defined above in this section. Then there exist functions $u, v \in L$ that satisfy the equation $B_Q(u,v) = 0$ and do not vanish identically in D .*

Proof: Obviously, the equation $u\Delta v + v\Delta u = 0$ will be satisfied if there exists a function p such that

$$\Delta u(x,y) + p(x,y)u(x,y) = 0, \quad (15)$$

$$\Delta v(x,y) - p(x,y)v(x,y) = 0 \quad (16)$$

for $(x,y) \in D$.

We look for solutions to (15)–(16) of a special form, namely, $u(x,y) = U(x)\tilde{U}(y)$ and $v(x,y) = V(x)\tilde{V}(y)$. In order to satisfy the boundary condition (14), we assume that

$$U\left(\pm\frac{\pi}{2}\right) = \tilde{U}\left(\pm\frac{\pi}{2}\right) = V\left(\pm\frac{\pi}{2}\right) = \tilde{V}\left(\pm\frac{\pi}{2}\right) = 0. \quad (17)$$

Moreover, we assume that p is a function of a single variable x , i. e., that p is independent of y . Under these assumptions, (15)–(16) yield

$$U''(x)\tilde{U}(y) + U(x)\tilde{U}''(y) + p(x)U(x)\tilde{U}(y) = 0,$$

$$V''(x)\tilde{V}(y) + V(x)\tilde{V}''(y) - p(x)V(x)\tilde{V}(y) = 0,$$

or, after separation of variables,

$$\frac{U''(x)}{U(x)} + p(x) = -\frac{\tilde{U}''(y)}{\tilde{U}(y)} = \lambda, \quad (18)$$

$$\frac{V''(x)}{V(x)} - p(x) = -\frac{\tilde{V}''(y)}{\tilde{V}(y)} = \mu, \quad (19)$$

where λ and μ are some real constants. For our purpose it is sufficient to find at least one non-zero solution to (18) and (19) satisfying the boundary condition (17). So, we put $\lambda = \mu = 1$ for simplicity. Then we easily check that $\tilde{U}(y) = \tilde{V}(y) = \cos y$ satisfy both the equations (18)–(19) and boundary conditions (17).

The problem of finding the functions $U(x)$, $V(x)$, and $p(x)$ satisfying the equations (18)–(19) and boundary conditions (17) is more complicated. We put by definition $a = -1$ and $p(x) = -2q \cos 2x$, where q is a real number that will be specified later. Then the equation (18) implies that the function $U(x)$ satisfies the equation

$$z''(x) + (a - 2q \cos 2x)z(x) = 0. \quad (20)$$

Similarly, the equation (19) implies that the function $V(x)$ satisfies the equation

$$z''(x) + (a + 2q \cos 2x)z(x) = 0, \quad (21)$$

which can be obtained from (20) by substituting $-q$ instead of q .

The equation (20) is known as the Mathieu equation. Properties of its solutions are studied in hundreds of articles and books among which we mention only three classical treatises [3], [4], and [5] and the chapters [6], [7] in well-known handbooks.

Recall from [3]–[7] that one of periodic solutions of the Mathieu equation (20) is usually denoted by $\text{se}_2(x, q)$ and may be defined by the following expansion

$$\text{se}_2(x, q) = \sum_{m=1}^{\infty} B_{2m}^2 \sin(2mx), \quad (22)$$

where the coefficients B_{2m}^2 satisfy the recurrence relations $qB_4^2 = (b_2 - 4)B_2^2$ and $qB_{2m+2}^2 = (b_2 - 4m^2)B_{2m}^2 - qB_{2m-2}^2$ for $m \geq 2$, as well as the following normalization condition $\sum_{m=1}^{\infty} (B_{2m}^2)^2 = 1$.

Recall also that, for a given value of q , the function $\text{se}_2(x, q)$ is a solution of the Mathieu equation (20) not for arbitrary values of the parameter a ; the parameter a , referred to as an eigenvalue, depends on q and, in fact, is a function of q . Usually, this function is denoted by $a = b_2(q)$. Though this function is quite complicated, it is known that $b_2(-q) = b_2(q)$ and there is $q_* \approx 8$ such that $b_2(q_*) = -1$.

From these properties of the function $\text{se}_2(x, q)$ we immediately conclude that the function $U(x) = \text{se}_2(x, q_*)$ satisfies the equation (20) and the boundary conditions (17). We conclude also that the function $V(x) = \text{se}_2(x, -q_*)$ satisfies the equation (21) and the boundary conditions (17).

Hence, the functions $u(x, y) = \text{se}_2(x, q_*) \cos y$ and $v(x, y) = \text{se}_2(x, -q_*) \cos y$ satisfy the equation $u\Delta v + v\Delta u = 0$ in D and vanish on ∂D . This completes the proof of Theorem 3.

Theorem 4: *Let D , L , and Q be as defined at the beginning of this section. Then there exist functions $u, v \in L$ that satisfy the equation $Q(u) = Q(v)$ and neither $u = v$, nor $u = -v$.*

Proof follows immediately from Theorems 2 and 3.

References

- [1] M. Adam, S. Czerwik, Quadratic operators and quadratic functional equation. In the book: P.M. Pardalos (ed.) et al., *Nonlinear analysis. Stability, approximation, and inequalities*. New York: Springer, 2012. P. 13–37. MR2962629, Zbl 1248.39020.
- [2] Xi-Nan Ma, Concavity estimates for a class of nonlinear elliptic equations in two dimensions. *Math. Z.* **240** (2002), no.1, 1–11. MR1906704 (2003h:35082), Zbl 1003.35047.
- [3] M.J.O. Strutt, *Lamésche-, Mathiesche- und verwandte Funktionen in Physik und Technik*. Berlin: Springer, 1932. MR0350085 (50 #2578), Zbl 0005.16005.
- [4] N.W. McLachlan, *Theory and applications of Mathieu functions*. Oxford: Clarendon Press, 1947. MR0021158 (9,31b), Zbl 0029.02901.
- [5] A. Angot, *Compléments de mathématiques à l'usage des ingénieurs de l'électrotechnique et des télécommunications*. Paris: Éditions de la Revue d'Optique, 1952. MR0049256 (14,145f), Zbl 0047.28401.
- [6] G. Blanch, Mathieu functions. In the book: M. Abramowitz, I.A. Stegun (eds.) *Handbook of mathematical functions with formulas, graphs and mathematical tables*. Washington: U.S. Department of Commerce, 1964. MR0167642 (29 #4914), Zbl 0171.38503.
- [7] G. Wolf, Mathieu functions and Hill's equation. In the book: F.W.J. Olver, D.W. Lozier, R.F. Boisvert, Ch.W. Clark (eds.), *NIST handbook of mathematical functions*. Cambridge: Cambridge University Press, 2010. See also online companion of this book at <http://dlmf.nist.gov>. MR2655368, Zbl 1198.00002.

Victor Alexandrov
Sobolev Institute of Mathematics
Koptug ave., 4
Novosibirsk, 630090, Russia
and
Department of Physics
Novosibirsk State University
Pirogov str., 2
Novosibirsk, 630090, Russia
e-mail: alex@math.nsc.ru

Submitted: June 8, 2015